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ON NON-STATIONARY MARKOV CHAINS WITH CONVERGING TRANSITION MATRICES

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On non-stationary Markov Chains with converging transition matrices *

by

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ABSTRACT

Recent papers have shown that $\Pi_{k=1}^{\infty} P(k) = \lim_{m\to\infty} (P(m)\ldots P(1))$ exists whenever the sequence of stochastic matrices $\{P(k)\}_{k=1}^{\infty}$ exhibits convergence to a unichained and aperiodic matrix P. We show how the limit matrix depends upon P(1).

In addition we prove that $\lim_{m\to\infty} \lim_{n\to\infty} (P(n+m)\ldots P(m+1))$ exists and equals the invariant probability matrix associated with P. The convergence rate is determined by the rate of convergence of $\{P(k)\}_{k=1}^{\infty}$ towards P.

KEY WORDS & PHRASES: *non-stationary Markov chains; backwards products; ergodicity; convergence rates; invariant probability matrix.*

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In two recent papers by ANTHONISSE and TIJMS [1], and CHATTERJEE and SENETA [3], the asymptotic behaviour was studied of *backwards* matrix products of the type

(1) $P(n)\ldots P(k) \quad \text{as } n \to \infty; \quad k = 1,2,\ldots$

where $\{P(m)\}_{m=1}^{\infty}$ is a non-stationary N-state Markov chain, with

(2) $\lim_{n\to\infty} P(n) = P.$

Matrix products of the type (1) are strongly related to the *forward* products, known as inhomogeneous Markov chains, and studied in an extensive literature that started with the papers by HAJNAL [7], (cf. [8],[10] and [11] for a survey of the present state of the art).

The backward matrix products arise e.g.

(a) in estimate modification processes, where n individuals each of whom has an estimate of some unknown quantity, enter information exchanges which lead them to readapt their estimates in an (infinite) sequence of iterations (cf. DE GROOT [5], and CHATTERJEE and SENETA [3] and DALKEY [4])

(b) in non-stationary Markov Decision Processes when analyzing the total reward in a planning period of n epochs as n tends to infinity (cf. MORTON and WECKER [9], and BOWERMAN [2])

(c) when applying value-iteration methods to Markov Decision Processes the transition probabilities of which are unknown in advance in the sense that only sequences of (converging) approximations can be obtained (cf. FEDERGRUEN and SCHWEITZER [6]).

Let U(r,k) be the stochastic matrix defined by

(3) $U(r,k) = P(r+k)\ldots P(r+1), \quad k = 1,2,\ldots .$

The sequence $\{P(k)\}_{k=1}^{\infty}$ is said to be ergodic (in a bakcwards direction) if

(4) $\lim_{k\to\infty} U(r,k) = \underline{1}D(r)', \quad r \geq 0$

where D(r) is obviously a probability vector, i.e. $D(r) \geq 0$ and $\sum_i D(r_i) = 1$. Ergodicity of $\{P(k)\}_{k=1}^{\infty}$ was shown in CHATTERJEE and SENETA [3] (th.5 and corollary) for the case where P is aperiodic and unichained and can equally be obtained by a mere adaptation of the proof of th.1 in ANTHONISSE and TIJMS [1]. Also in these papers the convergence in (4) was shown to be geometrical. Hence we have:

LEMMA 1. *Assume that* P *is unichained and aperiodic. Then* $\lim_{k\to\infty} U(r,k) = 1D(r)'$ *where there exist numbers* $M > 0$ *and* $0 < \lambda \leq 1$, *with*

$$(5) \qquad |U(r,k) - 1D(r)'| \leq M\lambda^k; \quad r,k = 1,2,\ldots . \quad \square$$

Note that the rate of convergence of $\{U(r,k)\}_{k=1}^{\infty}$ is independent of the rate at which $\{P(k)\}_{k=1}^{\infty}$ approaches P.

In this note we characterize the asymptotic behaviour of $\{D(r)\}_{r=1}^{\infty}$, as is especially required for the application mentioned under (c). First, however, example 1 shows that D(r) may heavily depend upon P(r), the first matrix in the product.

For any $N \times N$-stochastic matrix Q and for $j = 1,\ldots,N$ let

$$(6) \qquad M_j(Q) = \max_i Q_{ij} \quad \text{and} \quad m_j(Q) = \min_i Q_{ij}$$

and note from the identity $Q(2)Q(1)_{ij} = \sum_k Q(2)_{ik} Q(1)_{kj}$, that for any pair Q(1), Q(2) of stochastic matrices:

$$(7) \qquad M_j(Q(2)Q(1)) \leq M_j(Q(1)) \quad \text{and} \quad m_j(Q(2)Q(1)) \geq m_j(Q(1)); \; j = 1,\ldots,N.$$

A matrix is said to be strictly positive, if all of its entries are strictly positive.

EXAMPLE 1. In this example we show that D(r) is strictly positive whenever P(r) is. In other words, whenever $P(r) > 0$ and P has transient states, $D(r) \neq \pi$ where π is the (unique) stationary probability distribution associated with the matrix P.

To verify the implication $P(r) > 0 \Rightarrow D(r) > 0$, note from (1.7) that

$m_j(P(r)) \le m_j(U(r,k)) \le M_j(U(r,k)) \le M_j(P(r))$ for $r,k = 1,2,\dots$ and $j = 1,\dots,N$. Conclude that for all $i = 1,\dots,N$: $D(r)_j = \lim_{k\to\infty} U(r,k)_{ij} \ge m_j(P(r)) > 0$.

The next theorem shows that $1\pi'$ appears as the limit matrix when both r and k tend to infinity in the matrix product $U(r,k)$; in addition, the rate of convergence is specified.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $|P(k)-P| = \max_{ij} |P(k)_{ij}-P_{ij}| \le \varepsilon_k$.

THEOREM 2.

(8) $$\lim_{r\to\infty}\lim_{k\to\infty} U(r,k) = \lim_{r\to\infty} \underline{1}D(r)' = \underline{1}\pi',$$

and

(9) $$|D(r)-\pi| = O(\varepsilon_r).$$

PROOF. We first prove by complete induction with respect to k that

(10) $$|U(r,k)-P^k| = O(\varepsilon_r), \qquad r,k = 1,2,\dots .$$

Note that (1.10) holds for k = 1 and assume it holds for some k. Then,

$$|U(r,k+1)-P^{k+1}| = |P(r+k+1)U(r,k)-P(r+k+1)P^k+P(r+k+1)P^k-P^{k+1}|$$

$$\le O(\varepsilon_r) + |P(r+k+1)-P| \le O(\varepsilon_r) + \varepsilon_{r+k+1} \le O(\varepsilon_r).$$

Fix $j = 1,\dots,N$ and $\delta > 0$ and recall from the aperiodicity and unichainedness of P that there exists an integer $n \ge 1$ such that

$$P^n_{ij} - \delta \le \pi_j \le P^n_{ij} + \delta; \qquad i = 1,\dots,N.$$

Hence,

(11) $$M_j(P^n) - \delta \le \pi_j \le m_j(P^n) + \delta.$$

Use (10) with k = n and the fact that both $M_j(.)$ and $m_j(.)$ are Lipschitz continuous functions on the set of all N × N-matrices, to conclude that,

(12) $$|M_j(U(r,n))-M_j(P^n)| = O(\varepsilon_r); \; |m_j(U(r,n))-m_j(P^n)| = O(\varepsilon_r).$$

Insert (12) into (11) to conclude that

(13) $$M_j(U(r,n)) - O(\varepsilon_r) - \delta \le \pi_j \le m_j(U(r,n)) + O(\varepsilon_r) + \delta.$$

Next one verifies by a repeated application of (7) and in view of the fact that $M_j(.)$ and $m_j(.)$ are Lipschitz-continuous, that for all r = 1,2,... $\{M_j(U(r,k)\}_{k=1}^{\infty}$ and $\{m_j(U(r,k))\}_{k=1}^{\infty}$ are resp. monotonically non-increasing and non-decreasing towards $M_j(\underline{1}D(r)') = m_j(\underline{1}D(r)') = D(r)_j$. In particular we have for all r = 1,2,... :

(14) $$m_j(U(r,n)) \le D(r)_j \le M_j(U(r,n))$$

and insert (14) into (13) to conclude that for all $\delta > 0$

(15) $$D(r)_j - O(\varepsilon_r) - \delta \le \pi_j \le D(r)_j + O(\varepsilon_r) + \delta$$

and hence

$$|D(r)_j-\pi_j| = O(\varepsilon_r). \quad \square$$

Finally, example 2 below shows that the upperbound for the rate of convergence of $\{D(r)\}_{r=1}^{\infty}$ towards π is the sharpest possible one:

<u>EXAMPLE 2</u>. Let

$$P(k) = \begin{bmatrix} \frac{1}{2} + \alpha_k & \frac{1}{2} - \alpha_k \\ \frac{1}{2} + \alpha_k & \frac{1}{2} - \alpha_k \end{bmatrix} \text{ where } \{\alpha_k\}_{k=1}^{\infty} \downarrow 0.$$

Verify that U(r,k) = P(r) for all k = 1,2,..., such that $D(r) = \lim_{k\to\infty} U(r,k) = P(r)$. Conclude that $\{D(r)\}_{r=1}^{\infty}$ approaches π at the same rate as is exhibited by the convergence of $\{P(k)\}_{k=1}^{\infty}$ towards P (or alternatively by the rate of convergence of $\{\alpha_k\}_{k=1}^{\infty}$ towards zero).

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